Some Properties of Bipolar Fuzzy Normal HX Subgroup and its Normal Level Sub HX Groups

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Abstract

In this paper, the notion of the normal level sub HX group is introduced. We discuss the properties of normal level sub HX groups of a bipolar fuzzy normal HX subgroup of a HX group under homomorphism and anti-homomorphism. AMS Subject Classification (2000): 20N25, 03E72, 03F055, 06F35, 03G25.  
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I. INTRODUCTION

In 1965, Zadeh [23] introduced the notion of fuzzy subsets as a generalization of the notion of subsets in ordinary set theory. There were several attempts to fuzzily various mathematical structures. The fuzzifications of algebraic structures was initiated by Rosenfeld [21]. He introduced the notion of fuzzy subgroups and obtained some of their basic properties. Most of the recent works on fuzzy groups follow Rosenfeld’s definition. S.D.Kim et.al [8] investigated the concepts of Holomorphic images and preimage on fuzzy ideal. Homomorphic images and pre image of anti-fuzzy ideals are investigated by K.H.Him et.al [9]. The notion of anti-holomorphic images and pre image of fuzzy and anti-fuzzy ideals are investigated by K.C.Chandrasekara Rao et.al [3]. The concept of HX group was introduced by Li Hongxing [12] and the authors Luo Chengzhong, Mi Honghai, Li Hong Xing [13] introduced the concept of fuzzy HX group. In fuzzy set the membership degree of elements ranges over the interval [0, 1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval (0, 1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. Bipolar valued fuzzy set, which was introduced by K.M.Lee [11] are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree 1 indicates that elements somewhat satisfy the property and the membership degree [-1, 0) indicates that elements somewhat satisfy the implicit counter-property. S.V.Manemaran et.al [14] introduced the concept of bipolar fuzzy groups and fuzzy d-ideals of group under (T,S) norm and investigate several properties. R.Muthuraj et.al [17] introduced the concept of bipolar fuzzy HX subgroup and its level sub HX groups. R.Muthuraj et.al [18] introduced the concept of bipolar fuzzy normal HX subgroup. In this paper we discuss the properties of level sub HX groups under homomorphism and anti-homomorphism and also normal level sub HX groups under homomorphism and anti-homomorphism.

II. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, $G = (G,*)$ is a finite group, $e$ is the identity element of $G$, and $xy$ we mean $x * y$.

A. Definition 2.1 [23]
Let $S$ be a non-empty set. A fuzzy set $\mu$ on $S$ is a function $\mu: S \rightarrow [0,1]$.

B. Definition 2.2
Let $f$ be any function from a set $S$ to a set $T$, and let $\mu$ be any fuzzy subset of $S$. Then $\mu$ is called $f$-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, where $x,y \in S$. 
C. Definition 2.3
A fuzzy subset $\mu$ of $X$ is said to have sup property if, for any subset $A$ of $X$, there exist $a_0 \in A$ such that $\mu (a_0) = \max \{\mu(a) : a \in A\}$.

D. Definition 2.4
A fuzzy subset $\mu$ of $X$ is said to have inf property if, for any subset $A$ of $X$, there exist $a_0 \in A$ such that $\mu (a_0) = \min \{\mu(a) : a \in A\}$.

E. Definition 2.5 [13]
Let $G$ be a finite group. In $2^G - \{\phi\}$, a non-empty set $\mathcal{H} \subset 2^G - \{\phi\}$ is called a HX group of $G$, if $\mathcal{H}$ is a group with respect to the algebraic operation defined by $AB = \{ ab / a \in A \text{ and } b \in B \}$.

F. Definition 2.6 [16]
A mapping $f$ from a HX group $\mathcal{H}_1$ to a HX group $\mathcal{H}_2$ is said to be a homomorphism if $f(XY) = f(X)f(Y)$ for all $X, Y \in \mathcal{H}_1$.

G. Definition 2.7 [16]
A mapping $f$ from a HX group $\mathcal{H}_1$ to a HX group $\mathcal{H}_2$ ( $\mathcal{H}_1 \text{ and } \mathcal{H}_2$ are not necessarily commutative) is said to be an anti-homomorphism if $f(XY) = f(Y)f(X)$ for all $X, Y \in \mathcal{H}_1$.

H. Definition 2.8 [17]
Let $\mathcal{H}$ be a non-empty set. A bipolar-valued fuzzy set $\lambda_\mu$ in $\mathcal{H}$ is an object having the form $\lambda_\mu = \{(A, \lambda^+ \mu(A), \lambda^- \mu(A)) : A \in \mathcal{H} \}$ where $\lambda^+ \mu : \mathcal{H} \rightarrow [0,1]$ and $\lambda^- \mu : \mathcal{H} \rightarrow [-1,0]$ are mappings. The positive membership degree $\lambda^+ \mu(A)$ denotes the satisfaction degree of an element $A$ to the property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{(A, \lambda^+ \mu(A), \lambda^- \mu(A)) : A \in \mathcal{H} \}$, and the negative membership degree $\lambda^- \mu(A)$ denotes the satisfaction degree of an element $A$ to some implicit counter property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{(A, \lambda^+ \mu(A), \lambda^- \mu(A)) : A \in \mathcal{H} \}$. If $\lambda^+ \mu(A) = 0$ and $\lambda^- \mu(A) = 0$, it is the situation that $A$ is regarded as having only positive satisfaction for $\lambda_\mu = \{(A, \lambda^+ \mu(A), \lambda^- \mu(A)) : A \in \mathcal{H} \}$. If $\lambda^+ \mu(A) = 0$ and $\lambda^- \mu(A) = 0$, it is the situation that $A$ does not satisfy the property of $\lambda_\mu = \{(A, \lambda^+ \mu(A), \lambda^- \mu(A)) : A \in \mathcal{H} \}$, but somewhat satisfies the counter property of $\lambda_\mu = \{(A, \lambda^+ \mu(A), \lambda^- \mu(A)) : A \in \mathcal{H} \}$. It is possible for an element $A$ to be such that $\lambda^+ \mu(A) \neq 0$ and $\lambda^- \mu(A) \neq 0$ when the membership function of property overlaps that its counter property over some portion of $\mathcal{H}$. For the sake of simplicity, we shall use the symbol $\lambda_\mu = (\lambda^+ \mu, \lambda^- \mu)$ for the bipolar-valued fuzzy set $\lambda_\mu = \{(A, \lambda^+ \mu(A), \lambda^- \mu(A)) : A \in \mathcal{H} \}$.

I. Definition 2.9
Let $f$ be any function from a set $\mathcal{H}_1$ to a set $\mathcal{H}_2$, and let $\lambda_\mu$ be any bipolar fuzzy subset of $X$. Then $\lambda_\mu$ is called $f$-invariant if $f(X) = f(Y)$ implies $\lambda_\mu^+(X) = \lambda_\mu^+(Y)$ and $\lambda_\mu^-(X) = \lambda_\mu^-(Y)$, where $X, Y \in \mathcal{H}_1$.

J. Definition 2.10 [16]
Let $G_1$ and $G_2$ be any two groups. Let $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ be bipolar fuzzy subsets in $G_1$ and $G_2$ respectively. Let $\mathcal{H}_1 \subset 2^{G_1} - \{\phi\}$ and $\mathcal{H}_2 \subset 2^{G_2} - \{\phi\}$ be HX groups defined on $G_1$ and $G_2$ respectively. Let $\lambda_\mu = (\lambda^+ \mu, \lambda^- \mu)$ and $\sigma_\varphi = (\sigma^+ \varphi, \sigma^- \varphi)$ are bipolar fuzzy subsets defined on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively induced by $\mu$ and $\varphi$. Let $f : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a mapping, then the image $f(\lambda_\mu)$ of $\lambda_\mu$ is the bipolar fuzzy subset $f(\lambda_\mu) = (f(\lambda^+ \mu), f(\lambda^- \mu))$ of $\mathcal{H}_2$ defined by for all $f(X) = Y \in \mathcal{H}_2$, where $X \in \mathcal{H}_1$.

\[
(f(\lambda_\mu))^+(f(X)) = \begin{cases} \max \{\lambda_\mu^+(X) : X \in f^{-1}(Y)\} & \text{if } f^{-1}(Y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
(f(\lambda_\mu))^- (f(X)) = \begin{cases} \max \{\lambda_\mu^-(X) : X \in f^{-1}(Y)\} & \text{if } f^{-1}(Y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}
\]

also the pre-image $f^{-1}(\sigma_\varphi)$ of $\sigma_\varphi$ under $f$ is a bipolar fuzzy subset of $\mathcal{H}_1$ defined by $(f^{-1}(\sigma^+ \varphi))(X) = \sigma^+ \varphi(f(X))$, $(f^{-1}(\sigma^- \varphi))(X) = \sigma^- \varphi(f(X))$.

K. Definition 2.11 [17]
Let $\mu$ be a bipolar fuzzy subset defined on $G$. Let $\mathcal{H} \subset 2^G - \{\phi\}$ be a HX group of $G$. A bipolar fuzzy set $\lambda_\mu$ defined on $\mathcal{H}$ is said to be a bipolar fuzzy subgroup induced by $\mu$ on $\mathcal{H}$ or a bipolar fuzzy HX subgroup of $\mathcal{H}$ if for $X, Y \in \mathcal{H}$,

1) $\lambda_\mu^+(XY) \leq \min \{\lambda_\mu^+(X), \lambda_\mu^+(Y)\}$

2) $\lambda_\mu^+(XY) \leq \max \{\lambda_\mu^-(X), \lambda_\mu^-(Y)\}$

3) $\lambda_\mu^+(X^{-1}) = \lambda_\mu^-(X), \lambda_\mu^-(X^{-1}) = \lambda_\mu^+(X)$, where $\lambda_\mu^+(X) = \max \{\mu^+(x) \text{ for all } x \in X \subseteq G\}$.
\[ \lambda_n^+(X) = \min \{ \mu^+(x) / \text{for all } x \in X \subseteq G \}. \]

**L. Definition 2.12 [17]**

Let \( \lambda_n \) be a bipolar fuzzy HX subgroup of a HX group \( \vartheta \). The set \( U(\lambda_n ; \alpha, \beta) = \{ A \in \vartheta / \lambda_n^+(A) \geq \alpha, \lambda_n^-(A) \leq \beta \} \), for any \( < \alpha, \beta > \in [0,1] \times [0,1] \) is called the bipolar level subset of \( \lambda_n \) or \( < \alpha, \beta > \) level subset of \( \lambda_n \) or upper level subset of \( \lambda_n^+ \) or level subset of \( \lambda_n^+ \).

**M. Theorem 2.13 [17]**

Let \( \lambda_n \) be a bipolar fuzzy HX subgroup of HX group \( \vartheta \) then for \( < \alpha, \beta > \in [0,1] \times [0,1] \) such that \( \lambda_n^+(E) \geq \alpha, \lambda_n^-(E) \leq \beta \) and \( U(\lambda_n ; \alpha, \beta) \) is a sub HX group of \( \vartheta \).

**N. Theorem 2.14 [17]**

Let \( \vartheta \) be a HX group and \( \lambda_n \) be a bipolar fuzzy subset of \( \vartheta \) such that \( U(\lambda_n ; \alpha, \beta) \) is a sub HX group of \( \vartheta \) for \( < \alpha, \beta > \in [0,1] \times [0,1] \) such that \( \lambda_n^+(E) \geq \alpha \) and \( \lambda_n^-(E) \leq \beta \), then \( \lambda_n \) is a bipolar fuzzy HX group of \( \vartheta \).

**O. Definition 2.15 [17]**

Let \( \lambda_n \) be a bipolar fuzzy HX subgroup of a HX group \( \vartheta \). The sub HX groups \( U(\lambda_n ; \alpha, \beta) \) for \( < \alpha, \beta > \in [0,1] \times [0,1] \) and \( \lambda_n^+(E) \geq \alpha, \lambda_n^-(E) \leq \beta \) are called bipolar level sub HX groups of \( \lambda_n \) or level sub HX groups of \( \lambda_n \).

**P. Definition 2.16 [18]**

Let \( G \) be a group. Let \( \mu \) be a bipolar fuzzy subset defined on \( G \). Let \( \vartheta \subset 2^G - \{ \phi \} \) be a HX group on \( G \). Let \( \lambda_n = (\lambda_n^+, \lambda_n^-) \) be a bipolar fuzzy subset on \( \vartheta \), then \( \lambda_n \) is said to be bipolar fuzzy normal HX subgroup on \( \vartheta \), if
\[
\lambda_n^+(XYX^{-1}) \geq \lambda_n^+(Y), \lambda_n^-(XYX^{-1}) \leq \lambda_n^-(Y), \text{ for all } X, Y \in \vartheta, \text{ where,}
\]
\[
\lambda_n^+(X) = \max \{ \mu^+(x) / \text{for all } x \in X \subseteq G \} \text{ and}
\]
\[
\lambda_n^-(X) = \min \{ \mu^-(x) / \text{for all } x \in X \subseteq G \}.
\]

**Q. Theorem 2.17 [18]**

Let \( G \) be a group and \( \vartheta \subset 2^G - \{ \phi \} \) be a HX group on \( G \). Let \( \lambda_n = (\lambda_n^+, \lambda_n^-) \) be a bipolar fuzzy HX subgroup of a HX group \( \vartheta \), then the following conditions are equivalent
1) \( \lambda_n \) is a bipolar fuzzy normal HX subgroup on \( \vartheta \).
2) \( \lambda_n^+(XYX^{-1}) = \lambda_n^+(Y) \), for all \( X, Y \in \vartheta \).
3) \( \lambda_n^-(XY) = \lambda_n^-(YX) \), for all \( X, Y \in \vartheta \).

**R. Remark 2.18**

Let \( G \) be a group. Let \( \mu = (\mu^+, \mu^-) \) be a bipolar fuzzy subset of \( G \). If \( \lambda_n = (\lambda_n^+, \lambda_n^-) \) be a bipolar fuzzy normal HX subgroup of \( \vartheta \) with \( |X| \geq 2 \) for all \( X \in \vartheta \), then \( \mu = (\mu^+, \mu^-) \) need not be a bipolar fuzzy subgroup of \( G \). This can be illustrated by the following Example.

Let \( G = \{ Z_7 - \{ 0 \}, \ast \} \) be a group. For all \( x \in G \), Define a bipolar fuzzy set \( \mu = (\mu^+, \mu^-) \) on \( G \) as,

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu^+(x) )</td>
<td>0.8</td>
<td>0.6</td>
<td>0.5</td>
<td>0.6</td>
<td>0.5</td>
<td>0.4</td>
</tr>
<tr>
<td>( \mu^-(x) )</td>
<td>-0.7</td>
<td>-0.6</td>
<td>-0.4</td>
<td>-0.6</td>
<td>-0.4</td>
<td>-0.3</td>
</tr>
</tbody>
</table>

But, \( \mu^+(2 \cdot 3) = \mu^+(6) = 0.4 \)
Min \( \{ \mu^+(2), \mu^+(3) \} = \min \{ 0.6, 0.5 \} = 0.5 \)
Hence, \( \mu^+(2 \cdot 3) \geq \min \{ \mu^+(2), \mu^+(3) \} \)
And \( \mu^-(2 \cdot 3) = \mu^-(6) = -0.3 \)
Max \( \{ \mu^-(2), \mu^-(3) \} = \max \{ -0.6, -0.4 \} = -0.4 \)
Hence, \( \mu^-(2 \cdot 3) \leq \max \{ \mu^-(2), \mu^-(3) \} \)
So, \( \mu \) is not a bipolar fuzzy subgroup of \( G \).

Let \( \vartheta = \{ \{1,6\}, \{2,5\}, \{3,4\} \} \), then \( (\vartheta, \ast) \) be a HX subgroup of \( G \), where \( E = \{1,6\}, A = \{2,5\}, B = \{3,4\} \).

\[
\begin{array}{cccccc}
\ast & E & A & B \\
E & E & A & B \\
A & A & B & E \\
B & B & E & A \\
\end{array}
\]

For all \( X \in \vartheta \), Define a bipolar fuzzy set \( \lambda_n = (\lambda_n^+, \lambda_n^-) \) on \( \vartheta \) as,
\[
\lambda_n^+(X) = \max \{ \mu^+(x) / \text{for all } x \in X \subseteq G \}
\]
\[
\lambda_n^-(X) = \min \{ \mu^-(x) / \text{for all } x \in X \subseteq G \}
\]

Now,
\[
\lambda_n^+(E) = \lambda_n^+(\{1,6\}) = \max \{ \mu^+(1), \mu^+(6) \} = \max \{ 0.8, 0.4 \} = 0.8
\]
Let \( \lambda_\mu^+(x) = \lambda_\mu^-(2,5) = \max \{ \mu^+(2), \mu^+(5) \} = \max \{ 0.6, 0.5 \} = 0.6 \)

\( \lambda_\mu^-(B) = \lambda_\mu^-(3,4) = \max \{ \mu^-(3), \mu^-(4) \} = \max \{ 0.5, 0.6 \} = 0.6 \)

\( \lambda_\mu^-(E) = \lambda_\mu^-((1,6)) = \min \{ \mu^-((1), \mu^-((6)) = \min \{ -0.7, -0.3 \} = -0.7 \)

\( \lambda_\mu^-(A) = \lambda_\mu^-((2,5)) = \min \{ \mu^-((2), \mu^-((5)) = \min \{ -0.6, -0.4 \} = -0.6 \)

\( \lambda_\mu^- B) = \lambda_\mu^- ((3,4)) = \min \{ \mu^-((3), \mu^-((4)) = \min \{ -0.4, -0.6 \} = -0.6 \)

and also, \( \lambda_\mu^+ (XY) = \lambda_\mu^- (YX) \) \( \lambda_\mu^- (XY) = \lambda_\mu^- (YX) \) for all \( X, Y \in \mathcal{S} \).

By routine calculations, it is easy to see that \( \lambda_\mu \) is a bipolar fuzzy normal \( \mathcal{S} \) subgroup of \( \mathcal{S} \).

S. Theorem 2.19

Let \( G \) be a group and \( \mathcal{S} \subset 2^G - \{ \phi \} \) be a \( \mathcal{S} \) group on \( G \). Let \( \lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-) \) be a bipolar fuzzy normal \( \mathcal{S} \) subgroup of a \( \mathcal{S} \) group \( \mathcal{S} \), then \( \mathcal{H} = \{ X \in \mathcal{S} / \lambda_\mu^+(X) = \lambda_\mu^-(E) , \lambda_\mu^-(X) = \lambda_\mu^-(E) \} \) is a normal sub \( \mathcal{S} \) group of \( \mathcal{S} \).

1) Proof

Let \( \mathcal{H} = \{ X \in \mathcal{S} / \lambda_\mu^+(X) = \lambda_\mu^-(E) , \lambda_\mu^-(X) = \lambda_\mu^-(E) \} \).

Let \( \mathcal{H} \) be a non-empty subset of \( \mathcal{S} \), since \( E \in \mathcal{H} \).

By Theorem 3.3 (ii) [17], \( \mathcal{H} \) is a sub \( \mathcal{S} \) group of \( \mathcal{S} \).

For any \( A \in \mathcal{S} \) and \( X \in \mathcal{H} \),

Now, \( \lambda_\mu^+(AXA^{-1}) = \lambda_\mu^+(A^{-1}AX) \)
\( = \lambda_\mu^+(X) \)
\( \lambda_\mu^-(AXA^{-1}) = \lambda_\mu^-(X) \)
\( \lambda_\mu^- (AXA^{-1}) = \lambda_\mu^-(X) \).

Hence, \( \mathcal{H} \) is a normal sub \( \mathcal{S} \) group of \( \mathcal{S} \).

T. Theorem 2.20 [18]

Let \( G \) be a group. Let \( \lambda_\mu \) be a bipolar fuzzy \( \mathcal{S} \) subgroup of a \( \mathcal{S} \) group \( \mathcal{S} \) is a bipolar fuzzy normal \( \mathcal{S} \) subgroup on \( \mathcal{S} \) if and only if for any \( \alpha, \beta \in [0,1] \times [-1,0] \), \( U(\lambda_\mu ; \alpha, \beta) \) is a normal sub \( \mathcal{S} \) group of \( \mathcal{S} \).

U. Definition 2.21 [18]

Let \( \lambda_\mu \) be a bipolar fuzzy normal \( \mathcal{S} \) subgroup of a \( \mathcal{S} \) group \( \mathcal{S} \). The normal sub \( \mathcal{S} \) groups \( U(\lambda_\mu ; \alpha, \beta) \), for \( \alpha, \beta \in \mathcal{S} \), \( \lambda_\mu^+(E) \geq \alpha, \lambda_\mu^-(E) \leq \beta \) are called normal level sub \( \mathcal{S} \) groups of \( \lambda_\mu \).

III. Properties of Level Sub \( \mathcal{S} \) Groups of Bipolar Fuzzy \( \mathcal{S} \) Subgroup of a \( \mathcal{S} \) Group Under Homomorphism and Anti-Homomorphism

In this section, we investigate the properties of level sub \( \mathcal{S} \) groups of bipolar fuzzy \( \mathcal{S} \) subgroup of a \( \mathcal{S} \) group under homomorphism and anti-homomorphism.

A. Theorem 3.1

Let \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) be any two \( \mathcal{S} \) groups. \( f: \mathcal{S}_1 \rightarrow \mathcal{S}_2 \) be a mapping and \( U(\lambda_\mu ; \alpha, \beta) \) be a level sub \( \mathcal{S} \) group of a bipolar fuzzy \( \mathcal{S} \) subgroup of \( \mathcal{S}_1 \). Then \( U(f(\lambda_\mu ) ; \alpha, \beta) = f(U(\lambda_\mu ; \alpha, \beta)) \).

1) Proof

Let \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) be any two \( \mathcal{S} \) groups.

Let \( f: \mathcal{S}_1 \rightarrow \mathcal{S}_2 \) be a mapping.

Let \( U(\lambda_\mu ; \alpha, \beta) \) be a level sub \( \mathcal{S} \) group of a bipolar fuzzy \( \mathcal{S} \) subgroup of \( \mathcal{S}_1 \).

Let \( X \in U(\lambda_\mu ; \alpha, \beta) \) such that \( f(X) \in U(f(\lambda_\mu ) ; \alpha, \beta) \), where \( \alpha, \beta \in [0,1] \times [-1,0] \).

Let \( f(X) \in U(f(\lambda_\mu ) ; \alpha, \beta) \)
\( \iff (f(\lambda_\mu ))^+f(X) \geq \alpha \) and \( (f(\lambda_\mu ))^-f(X) \leq \beta \)
\( \iff \lambda_\mu^+(X) \geq \alpha \) and \( \lambda_\mu^-(X) \leq \beta \)
\( \iff X \in U(\lambda_\mu ; \alpha, \beta) \)
\( \iff f(X) \in U(U(\lambda_\mu ; \alpha, \beta)) \)

Hence, \( U(f(\lambda_\mu ) ; \alpha, \beta) = f(U(\lambda_\mu ; \alpha, \beta)) \).

B. Theorem 3.2

Let \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) be any two \( \mathcal{S} \) groups. Let \( f: \mathcal{S}_1 \rightarrow \mathcal{S}_2 \) be a mapping and \( U(\sigma_\phi ; \alpha, \beta) \) be a level sub \( \mathcal{S} \) group of a bipolar fuzzy \( \mathcal{S} \) subgroup of \( \mathcal{S}_2 \). Then \( U((f^{-1}(\sigma_\phi)) ; \alpha, \beta) = f^{-1}(U(\sigma_\phi ; \alpha, \beta)) \).
2) **Proof** 
Let $\vartheta_1$ and $\vartheta_2$ be any two HX groups. 
Let $f: \vartheta_1 \to \vartheta_2$ be a mapping. 
Let $U(\sigma_\vartheta; \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup of $\vartheta_2$. 
Let $X \in U(f^{-1}(\sigma_\vartheta); \alpha, \beta)$ \iff $(f^{-1}(\sigma_\vartheta))^+ (X) \geq \alpha$ and $(f^{-1}(\sigma_\vartheta))^- (X) \leq \beta$ 
\quad \iff $\sigma_\vartheta^+(f(X)) \geq \alpha$ and $\sigma_\vartheta^-(f(X)) \leq \beta$ 
\quad \iff $f(X) \in U(\sigma_\vartheta; \alpha, \beta)$ 
\quad \iff $X \in f^{-1}(U(\sigma_\vartheta; \alpha, \beta))$

Hence, $U(f^{-1}(\sigma_\vartheta); \alpha, \beta) = f^{-1}(U(\sigma_\vartheta; \alpha, \beta))$.

**C. Theorem 3.3** 
Let $G_1$ and $G_2$ be any two groups and $\vartheta_1$ and $\vartheta_2$ be HX groups on $G_1$ and $G_2$ respectively. Let $\lambda_\vartheta$ be a bipolar fuzzy HX subgroup on $\vartheta_1$. If $f: \vartheta_1 \to \vartheta_2$ is a homomorphism and onto, then the image of a level sub HX group $U(\lambda_\vartheta; \alpha, \beta)$ of a bipolar fuzzy HX subgroup $\lambda_\vartheta$ of a HX group $\vartheta_1$ is a level sub HX group $U(f(\lambda_\vartheta); \alpha, \beta)$ of a bipolar fuzzy HX subgroup $f(\lambda_\vartheta)$ of a HX group $\vartheta_2$.

3) **Proof**
Let $G_1$ and $G_2$ be any two groups and $f: \vartheta_1 \to \vartheta_2$ be a homomorphism. 
Let $\lambda_\vartheta$ be a bipolar fuzzy HX subgroup of a HX group $\vartheta_1$. 
Clearly, $f(\lambda_\vartheta)$ is a bipolar fuzzy HX subgroup of a HX group $\vartheta_2$. 
Let $X$ and $Y$ in $\vartheta_1$, implies $f(X)$ and $f(Y)$ in $\vartheta_2$. 
Let $U(\lambda_\vartheta; \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup $\lambda_\vartheta$ of a HX group $\vartheta_1$. 
Choose $\alpha$ and $\beta$ in such a way that $X, Y \in U(\lambda_\vartheta; \alpha, \beta)$ and hence $XY^{-1} \in U(\lambda_\vartheta; \alpha, \beta)$. 
Then, $\lambda_\vartheta^+(X) \geq \alpha$ and $\lambda_\vartheta^-(X) \leq \beta$ and $\lambda_\vartheta^+(Y) \geq \alpha$ and $\lambda_\vartheta^-(Y) \leq \beta$. 
Also $\lambda_\vartheta^+(XY^{-1}) \geq \alpha$ and $\lambda_\vartheta^-(XY^{-1}) \leq \beta$.

We have to prove that $U(f(\lambda_\vartheta); \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f(\lambda_\vartheta)$ of a HX group $\vartheta_2$.

Now, Let $f(X), f(Y) \in U(f(\lambda_\vartheta); \alpha, \beta)$. 
\quad \iff $(f(\lambda_\vartheta))^+(f(X)) = \lambda_\vartheta^+(X) \geq \alpha$, implies that $(f(\lambda_\vartheta))^+(f(X)) \geq \alpha$ 
\quad \iff $(f(\lambda_\vartheta))^-(f(Y)) = \lambda_\vartheta^-(Y) \geq \alpha$, implies that $(f(\lambda_\vartheta))^-(f(Y)) \geq \alpha$ and 
\quad \iff $(f(\lambda_\vartheta))^+(f(Y)) = \lambda_\vartheta^+(Y) \geq \beta$, implies that $(f(\lambda_\vartheta))^+(f(Y)) \leq \beta$. 
\quad \iff $(f(\lambda_\vartheta))^-(f(Y)) = \lambda_\vartheta^-(Y) \leq \beta$, implies that $(f(\lambda_\vartheta))^-(f(Y)) \leq \beta$.

Let, 
\quad $(f(\lambda_\vartheta))^+(f(X)(f(Y))^{-1}) = (f(\lambda_\vartheta))^+(f(X)f(Y^{-1}))$, as $f$ is a homomorphism 
\quad $= (f(\lambda_\vartheta))^+(f(X)f(Y^{-1}))$, as $f$ is a homomorphism 
\quad $= \lambda_\vartheta^+(XY^{-1})$ 
\quad $(f(\lambda_\vartheta))^-(f(X)(f(Y))^{-1}) \geq \alpha$ and 
\quad $(f(\lambda_\vartheta))^-(f(X)(f(Y))^{-1}) = (f(\lambda_\vartheta))^-(f(X)f(Y^{-1}))$, as $f$ is a homomorphism 
\quad $= (f(\lambda_\vartheta))^-(f(X)f(Y^{-1}))$, as $f$ is a homomorphism 
\quad $= \lambda_\vartheta^-(XY^{-1})$ 
\quad $\leq \beta$ 

That is, $(f(\lambda_\vartheta))^+(f(X)(f(Y))^{-1}) \geq \alpha$ and $(f(\lambda_\vartheta))^-(f(X)(f(Y))^{-1}) \leq \beta$. 
Hence, $(f(X)(f(Y))^{-1} \in U(f(\lambda_\vartheta); \alpha, \beta)$. 

Hence, $U(f(\lambda_\vartheta); \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f(\lambda_\vartheta)$ of a HX group $\vartheta_2$.

**D. Theorem 3.4**
Let $G_1$ and $G_2$ be any two groups and $\vartheta_1$ and $\vartheta_2$ be HX groups on $G_1$ and $G_2$ respectively. Let $\sigma_\vartheta$ be a bipolar fuzzy HX subgroup on $\vartheta_2$. If $f: \vartheta_1 \to \vartheta_2$ is a homomorphism, then the pre-image of a level sub HX group $U(\sigma_\vartheta; \alpha, \beta)$ of a bipolar fuzzy HX subgroup $\sigma_\vartheta$ of a HX group $\vartheta_2$ is a level sub HX group $U(f^{-1}(\sigma_\vartheta); \alpha, \beta)$ of a bipolar fuzzy HX subgroup $f^{-1}(\sigma_\vartheta)$ of a HX group $\vartheta_1$.

1) **Proof**
Let $\vartheta_1$ and $\vartheta_2$ be any two HX groups. 
Let $f: \vartheta_1 \to \vartheta_2$ be a homomorphism. 
Let $\sigma_\vartheta$ be a bipolar fuzzy HX subgroup of $\vartheta_2$. 
Clearly, $f^{-1}(\sigma_\vartheta)$ is a bipolar fuzzy HX subgroup of $\vartheta_1$. 
Let $X$ and $Y$ in $\vartheta_1$, implies $f(X)$ and $f(Y)$ in $\vartheta_2$. 
Let $U(\sigma_\vartheta; \alpha, \beta)$ be a level sub HX group of a bipolar fuzzy HX subgroup $\sigma_\vartheta$ of a HX group $\vartheta_2$. 
Choose $\alpha$ and $\beta$ in such a way that $f(X), f(Y) \in U(\sigma_\vartheta; \alpha, \beta)$ and hence $f(X)(f(Y))^{-1} \in U(\sigma_\vartheta; \alpha, \beta)$.
Then $\sigma_\alpha^-(f(X)) \geq \alpha$ and $\sigma_{\beta}^-(f(X)) \leq \beta$ and $\sigma_\alpha^+(f(Y)) \geq \alpha$ and $\sigma_{\beta}^-(f(Y)) \leq \beta$.

Also $\sigma_\alpha^+(f(X)) (f(Y)) \geq \alpha$ and $\sigma_{\beta}^-(f(X)) (f(Y)) \leq \beta$.

We have to prove that $U(f^{-1}(\sigma_\alpha) ; \alpha, \beta) \text{ is a level sub HX group of a bipolar fuzzy HX subgroup } f^{-1}(\sigma_\alpha) \text{ of a HX group } \mathcal{G}$.

Now, Let $X, Y \in U(f^{-1}(\sigma_\alpha) ; \alpha, \beta)$.

$(f^{-1}(\sigma_\alpha))^+(X) = \sigma_\alpha^+(f(X)) \geq \alpha$, implies that $(f^{-1}(\sigma_\alpha))^+(X) \geq \alpha$

$(f^{-1}(\sigma_\alpha))^+(Y) = \sigma_\alpha^+(f(Y)) \geq \alpha$, implies that $(f^{-1}(\sigma_\alpha))^+(Y) \geq \alpha$ and

$(f^{-1}(\sigma_\alpha))(X) = \sigma_{\beta}^-(f(X)) \leq \beta$, implies that $(f^{-1}(\sigma_\alpha))(X) \leq \beta$

$(f^{-1}(\sigma_\alpha))(Y) = \sigma_{\beta}^-(f(Y)) \leq \beta$, implies that $(f^{-1}(\sigma_\alpha))(Y) \leq \beta$.

Let,

$(f^{-1}(\sigma_\alpha))^+(X^Y \leq \beta)$

That is, $(f^{-1}(\sigma_\alpha))^+(X^Y \geq \alpha$ and $(f^{-1}(\sigma_\alpha))^+(X^Y \leq \beta)$.

Hence, $XY^\alpha \in U(f^{-1}(\sigma_\alpha) ; \alpha, \beta)$

Hence, $(f^{-1}(\sigma_\alpha) ; \alpha, \beta) \text{ is a level sub HX group of a bipolar fuzzy HX subgroup } f^{-1}(\sigma_\alpha) \text{ of a HX group } \mathcal{G}$.

---

**E. Theorem 3.5**

Let $G_1$ and $G_2$ be any two groups and $\mathcal{G}_1$ and $\mathcal{G}_2$ be HX groups on $G_1$ and $G_2$ respectively. Let $\lambda_\alpha \text{ be a bipolar fuzzy HX subgroup on } \mathcal{G}_1$. If $f: \mathcal{G}_1 \rightarrow \mathcal{G}_2 \text{ is an anti-homomorphism and onto, then the image of a level sub HX group } U(\lambda_\alpha ; \alpha, \beta) \text{ of a bipolar fuzzy HX subgroup } \lambda_\alpha \text{ of a HX group } \mathcal{G}_1 \text{ is a level sub HX group } U(f(\lambda_\alpha) ; \alpha, \beta) \text{ of a bipolar fuzzy HX subgroup } f(\lambda_\alpha) \text{ of a HX group } \mathcal{G}_2$.

1) **Proof**

Let $G_1$ and $G_2$ be any two groups and $f: \mathcal{G}_1 \rightarrow \mathcal{G}_2 \text{ be an anti-homomorphism}$.

Let $\lambda_\alpha \text{ be a bipolar fuzzy HX subgroup of a HX group } \mathcal{G}_1$.

Clearly, $f(\lambda_\alpha)$ is a bipolar fuzzy HX subgroup of a HX group $\mathcal{G}_2$.

Let $X$ and $Y$ in $\mathcal{G}_1$, implies $f(X)$ and $f(Y)$ in $\mathcal{G}_2$.

Let $U(\lambda_\alpha ; \alpha, \beta) \text{ be a level sub HX group of a bipolar fuzzy HX subgroup } \lambda_\alpha \text{ of a HX group } \mathcal{G}_1$.

Choose $\alpha$ and $\beta$ in such a way that $X, Y \in U(\lambda_\alpha ; \alpha, \beta)$ and hence $Y^\alpha X \in U(\lambda_\alpha; \alpha, \beta)$.

then $\lambda_\alpha^+(X) \geq \alpha$ and $\lambda_\alpha^+(X) \leq \beta$, $\lambda_\alpha^+(Y) \geq \alpha$ and $\lambda_\alpha^+(Y) \leq \beta$.

Also $\lambda_\alpha^+(X^Y) \geq \alpha$ and $\lambda_\alpha^-(X^Y) \leq \beta$.

We have to prove that $U(f(\lambda_\alpha) ; \alpha, \beta) \text{ is a level sub HX group of a bipolar fuzzy HX subgroup } f(\lambda_\alpha) \text{ of a HX group } \mathcal{G}_2$.

Now, Let $f(X), f(Y) \in U(f(\lambda_\alpha) ; \alpha, \beta)$.

$(f(\lambda_\alpha))^+(f(X)) = \lambda_\alpha^+(f(X)) \geq \alpha$, implies that $(f(\lambda_\alpha))^+(f(X)) \geq \alpha$

$(f(\lambda_\alpha))^+(f(Y)) = \lambda_\alpha^+(f(Y)) \geq \alpha$, implies that $(f(\lambda_\alpha))^+(f(Y)) \geq \alpha$ and

$(f(\lambda_\alpha))^-(f(X)) = \lambda_\alpha^-(f(X)) \leq \beta$, implies that $(f(\lambda_\alpha))^-(f(X)) \leq \beta$

$(f(\lambda_\alpha))^-(f(Y)) = \lambda_\alpha^-(f(Y)) \leq \beta$, implies that $(f(\lambda_\alpha))^-(f(Y)) \leq \beta$.

Let,

$(f(\lambda_\alpha))^+(f(X)(f(Y))^{-1}) = (f(\lambda_\alpha))^+(f(X)f(Y)^{-1})$, as $f$ is an anti-homomorphism

$= (f(\lambda_\alpha))^+(f(Y^{-1}X))$, as $f$ is an anti-homomorphism

$= \lambda_\alpha^+(f(Y^{-1}X))$

$(f(\lambda_\alpha))^+(f(X)(f(Y))^{-1}) \geq \alpha$, implies that $(f(\lambda_\alpha))^+(f(X)(f(Y))^{-1}) \geq \alpha$ and

$(f(\lambda_\alpha))^-(f(X)(f(Y))^{-1}) = (f(\lambda_\alpha))^-(f(X)(f(Y))^{-1})$, as $f$ is an anti-homomorphism

$= (f(\lambda_\alpha))^-(f(Y^{-1}X))$, as $f$ is an anti-homomorphism

$= \lambda_\alpha^-(f(Y^{-1}X)) \leq \beta$

That is, $(f(\lambda_\alpha))^+(f(X)(f(Y))^{-1}) \geq \alpha$ and $(f(\lambda_\alpha))^-(f(X)(f(Y))^{-1}) \leq \beta$.

Hence, $f(X)(f(Y))^{-1} \in U(f(\lambda_\alpha) ; \alpha, \beta)$.

Hence, $U(f(\lambda_\alpha) ; \alpha, \beta)$ is a level sub HX group of a bipolar fuzzy HX subgroup $f(\lambda_\alpha)$ of a HX group $\mathcal{G}_2$.  

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F. Theorem 3.6
Let $G_1$ and $G_2$ be any two groups and $\mathcal{H}_1$ and $\mathcal{H}_2$ be $H$-groups on $G_1$ and $G_2$ respectively. Let $\sigma_{\alpha}$ be a bipolar fuzzy $H$-subgroup on $\mathcal{H}_2$. If $f: \mathcal{H}_1 \to \mathcal{H}_2$ is an anti-homomorphism. Then the pre-image of a level sub $H$-group $U(\sigma_{\alpha}; \alpha, \beta) \subseteq \mathcal{H}_2$ is a level sub $H$-group $f^{-1}(U(\sigma_{\alpha}; \alpha, \beta))$ of $\mathcal{H}_1$.

1) Proof
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be any two $H$-groups.
Let $f: \mathcal{H}_1 \to \mathcal{H}_2$ be an anti-homomorphism.
Let $\sigma_{\alpha}$ be a bipolar fuzzy $H$-subgroup of $\mathcal{H}_2$.
Clearly, $f^{-1}(\sigma_{\alpha})$ is a bipolar fuzzy $H$-subgroup of $\mathcal{H}_1$.
Let $X$ and $Y$ in $\mathcal{H}_1$, implies $f(X)$ and $f(Y)$ in $\mathcal{H}_2$.
Let $U(\sigma_{\alpha}; \alpha, \beta)$ be a level sub $H$-group of a bipolar fuzzy $H$-subgroup $\sigma_{\alpha}$ of a $H$-group $\mathcal{H}_2$.
Choose $\alpha$ and $\beta$ in such a way that $f(X), f(Y) \in U(\sigma_{\alpha}; \alpha, \beta)$ and hence $(f(Y))^{-1}f(X) \in U(\sigma_{\alpha}; \alpha, \beta)$.

\[
\begin{align*}
\sigma_{d}(f(X)) & \geq \alpha \quad \text{and} \quad \sigma_{d}(f(Y)) \leq \beta, \quad \text{implies that} \quad (f^{-1}(\sigma_{\alpha}))^{*}(f(X)) \geq \alpha \\
(f^{-1}(\sigma_{\alpha}))^{*}(f(Y)) & \geq \alpha \quad \text{and} \quad (f^{-1}(\sigma_{\alpha}))^{*}(f(X)) \leq \beta, \quad \text{implies that} \quad (f^{-1}(\sigma_{\alpha}))^{*}(f(Y)) \leq \beta.
\end{align*}
\]

Let, $(f^{-1}(\sigma_{\alpha}))^{+}(XY^{-1}) = \sigma_{d}(f(XY^{-1}))$, as $f$ is an anti-homomorphism

\[
\begin{align*}
\sigma_{d}(f(XY^{-1})) & = \sigma_{d}(f(Y^{-1})f(X)) \quad \text{as} \quad f \quad \text{is an anti-homomorphism} \\
\sigma_{d}(f(XY^{-1})) & \geq \alpha \quad \text{and} \quad \sigma_{d}(f(XY^{-1})) \leq \beta.
\end{align*}
\]

Hence, $(f^{-1}(\sigma_{\alpha}))^{*}(XY^{-1}) \subseteq U(\sigma_{\alpha}; \alpha, \beta)$.

\[
\begin{align*}
U(\sigma_{\alpha}; \alpha, \beta) \subseteq U(f^{-1}(\sigma_{\alpha}); \alpha, \beta).
\end{align*}
\]

IV. Properties of Normal Level Sub $H$-Group of Bipolar Fuzzy Normal $H$-Subgroup of a $H$-Group Under Homomorphism and Anti-Homomorphism

In this section, we investigate the properties of normal level sub $H$-groups of bipolar fuzzy normal $H$-subgroup of a $H$-group under homomorphism and anti-homomorphism.

A. Theorem 4.1
Let $G_1$ and $G_2$ be any two groups and $\mathcal{H}_1$ and $\mathcal{H}_2$ be $H$-groups on $G_1$ and $G_2$ respectively. Let $\lambda_{\alpha}$ be a bipolar fuzzy normal $H$-subgroup on $\mathcal{H}_1$. If $f: \mathcal{H}_1 \to \mathcal{H}_2$ is a homomorphism and onto, then the image of a normal level sub $H$-group $U(\lambda_{\alpha}; \alpha, \beta)$ of a bipolar fuzzy normal $H$-subgroup $\lambda_{\alpha}$ is a level sub $H$-group $U(f(\lambda_{\alpha}); \alpha, \beta)$ of a bipolar fuzzy normal $H$-subgroup $f(\lambda_{\alpha})$ of a $H$-group $\mathcal{H}_2$.

1) Proof
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be any two $H$-groups.
Let $f: \mathcal{H}_1 \to \mathcal{H}_2$ be a homomorphism.
Let $\lambda_{\alpha}$ be a bipolar fuzzy normal $H$-subgroup of $\mathcal{H}_1$.
Clearly, $f(\lambda_{\alpha})$ is a bipolar fuzzy normal $H$-subgroup of $\mathcal{H}_2$.
Let $U(\lambda_{\alpha}; \alpha, \beta)$ be a normal level sub $H$-group of a bipolar fuzzy normal $H$-subgroup of $\mathcal{H}_1$.
Since $f$ is a homomorphism,
By Theorem 3.3, $U(f(\lambda_{\alpha}); \alpha, \beta)$ is a level sub $H$-group of a bipolar fuzzy $H$-subgroup $f(\lambda_{\alpha})$ of $\mathcal{H}_2$. 
Let $X \in \mathcal{D}$, $A \in U(\lambda_\alpha; \alpha, \beta)$ then $XAX^{-1} \in U(\lambda_\alpha; \alpha, \beta)$, since $U(\lambda_\alpha; \alpha, \beta)$ is normal level sub HX group.

Let, $f(X) \in \mathcal{D}$, $f(A) \in U(f(\lambda_\alpha); \alpha, \beta)$

Now, $f(X)f(A)(f(X))^{-1} = f(X)f(A)f(X^{-1})$

$= f(XAX^{-1})
\in f(U(\lambda_\alpha; \alpha, \beta))$, as $f$ is homomorphism.

Clearly, $f(X)f(A)(f(X))^{-1} \in U(f(\lambda_\alpha); \alpha, \beta)$. (By Theorem 3.1)

Hence, $U(f(\lambda_\alpha); \alpha, \beta))$ is a normal level sub HX group of a bipolar fuzzy normal HX subgroup $f(\lambda_\alpha)$ of $\mathcal{D}$.

### B. Theorem 4.2

Let $G_1$ and $G_2$ be any two groups and $\mathcal{D}_1$ and $\mathcal{D}_2$ be HX groups on $G_1$ and $G_2$ respectively. Let $\sigma_\alpha$ be a bipolar fuzzy normal HX subgroup on $\mathcal{D}_2$. If $f: \mathcal{D}_1 \to \mathcal{D}_2$ is a homomorphism, then the pre-image of a normal level sub HX group $U(\sigma_\alpha; \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup $\sigma_\alpha$ of a HX group $\mathcal{D}_2$ is a normal level sub HX group $U(f^{-1}(\sigma_\alpha); \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup $f^{-1}(\sigma_\alpha)$ of a HX group $\mathcal{D}_1$.

1) Proof

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be any two HX groups.

Let $f: \mathcal{D}_1 \to \mathcal{D}_2$ be a homomorphism.

Let $\sigma_\alpha$ be a bipolar fuzzy normal HX subgroup of $\mathcal{D}_2$.

Clearly, $f^{-1}(\sigma_\alpha)$ is a bipolar fuzzy normal HX subgroup of $\mathcal{D}_1$.

Let $U(\sigma_\alpha; \alpha, \beta)$ be a normal level sub HX group of a bipolar fuzzy normal HX subgroup $\sigma_\alpha$ of $\mathcal{D}_2$.

Since $f$ is a homomorphism,

By Theorem 3.4, $U(f^{-1}(\sigma_\alpha); \alpha, \beta))$ is a level sub HX group.

Let $X \in \mathcal{D}_1$, $A \in U(f^{-1}(\sigma_\alpha); \alpha, \beta)$

As, $U(\sigma_\alpha; \alpha, \beta)$ be a normal level sub HX group then $f(X) \in \mathcal{D}_2$, $f(A) \in U(\sigma_\alpha; \alpha, \beta)$

Such that, $f(X)f(A)(f(X))^{-1} \in U(\sigma_\alpha; \alpha, \beta)$

$= f(X)f(A)f(X^{-1}) \in U(\sigma_\alpha; \alpha, \beta)$

$= f(XAX^{-1}) \in U(\sigma_\alpha; \alpha, \beta)$, as $f$ is a homomorphism

$= XAX^{-1} \in f^{-1}(U(\sigma_\alpha; \alpha, \beta))$

Clearly, $XAX^{-1} \in U(f^{-1}(\sigma_\alpha); \alpha, \beta))$. (By Theorem 3.2)

Hence, $U(f^{-1}(\sigma_\alpha); \alpha, \beta)$ is a normal level sub HX group of a bipolar fuzzy normal HX subgroup $f^{-1}(\sigma_\alpha)$ of $\mathcal{D}_1$.

### C. Theorem 4.3

Let $G_1$ and $G_2$ be any two groups and $\mathcal{D}_1$ and $\mathcal{D}_2$ be HX groups on $G_1$ and $G_2$ respectively. Let $\lambda_\alpha$ be a bipolar fuzzy normal HX subgroup on $\mathcal{D}_1$. If $f: \mathcal{D}_1 \to \mathcal{D}_2$ is an anti-homomorphism and onto, then the image of a normal level sub HX group $U(\lambda_\alpha; \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup $\lambda_\alpha$ of a HX group $\mathcal{D}_1$ is a normal level sub HX group $U(f(\lambda_\alpha); \alpha, \beta)$ of a bipolar fuzzy normal HX subgroup $f(\lambda_\alpha)$ of a HX group $\mathcal{D}_2$.

1) Proof

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be any two HX groups.

Let $f: \mathcal{D}_1 \to \mathcal{D}_2$ be an anti-homomorphism.

Let $\lambda_\alpha$ be a bipolar fuzzy normal HX subgroup of $\mathcal{D}_1$.

Clearly, $f(\lambda_\alpha)$ is a bipolar fuzzy normal HX subgroup of $\mathcal{D}_2$.

Let $U(\lambda_\alpha; \alpha, \beta)$ be a normal level sub HX group of a bipolar fuzzy normal HX subgroup $f(\lambda_\alpha)$ of $\mathcal{D}_2$.

Since, $f$ is an anti-homomorphism,

By Theorem 3.3, $U(f(\lambda_\alpha); \alpha, \beta)$ is a level sub HX group of $\mathcal{D}_2$.

Let $X \in \mathcal{D}_1$, $A \in U(\lambda_\alpha; \alpha, \beta)$ then $X^{-1}AX \in U(\lambda_\alpha; \alpha, \beta)$, since $U(\lambda_\alpha; \alpha, \beta)$ be a normal level sub HX group.

Let, $f(X) \in \mathcal{D}_2$, $f(A) \in U(f(\lambda_\alpha); \alpha, \beta)$

Now, $f(X)f(A)(f(X))^{-1} = f(X)f(A)f(X^{-1})$

$= f(XAX^{-1})$

$\in f(U(\lambda_\alpha; \alpha, \beta))$ as $f$ is an anti-homomorphism

Clearly, $f(X)f(A)(f(X))^{-1} \in U(f(\lambda_\alpha); \alpha, \beta))$. (By Theorem 3.1)

Hence, $U(f(\lambda_\alpha); \alpha, \beta)$ is a normal level sub HX group of a bipolar fuzzy normal HX subgroup $f(\lambda_\alpha)$ of $\mathcal{D}_2$.

### D. Theorem 4.4

Let $G_1$ and $G_2$ be any two groups and $\mathcal{D}_1$ and $\mathcal{D}_2$ be HX groups on $G_1$ and $G_2$ respectively. Let $\sigma_\alpha$ be a bipolar fuzzy normal HX subgroup on $\mathcal{D}_2$. If $f: \mathcal{D}_1 \to \mathcal{D}_2$ is an anti-homomorphism and onto, then the pre-image of a normal level sub HX group
U(σ;α,β) of a bipolar fuzzy normal HX subgroup σ of a HX group Θ is a normal level sub HX group U(f⁻¹(σ);α,β) of a bipolar fuzzy normal HX subgroup f⁻¹(σ) of a HX group Θ₁.

1) Proof
Let Θ₁ and Θ₂ be any two HX groups.
Let f: Θ₁ → Θ₂ be an anti-homomorphism.
Let σ be a bipolar fuzzy normal HX subgroup of Θ₂.
Clearly, f⁻¹(σ) is a bipolar fuzzy normal HX subgroup of Θ₁.
Let U(σ;α,β) be a normal level sub HX group of a bipolar fuzzy normal HX subgroup σ of Θ₂.
By Theorem 3.4, U(f⁻¹(σ);α,β) is a level sub HX group.
Let X ∈ Θ₁, A ∈ U(f⁻¹(σ);α,β)
As, U(σ;α,β) be a normal level sub HX group then f(X) ∈ Θ₂, f(A) ∈ U(σ;α,β)
Such that,
(f(X⁻¹))(f(A))(f(X) ∈ U(σ;α,β)
=f(X⁻¹)f(A)f(X) ∈ U(σ;α,β)
=XAX⁻¹ ∈ f⁻¹(U(σ;α,β))
Clearly, XAX⁻¹ ∈ U(f⁻¹(σ);α,β). (By Theorem 3.2)

Hence, U(f⁻¹(σ);α,β) is a normal level sub HX group of a bipolar fuzzy normal HX subgroup f⁻¹(σ) of Θ₁.

V. CONCLUSION

In this paper, we introduced the notion of the normal level sub HX group of a bipolar fuzzy normal HX subgroup and studied this structure. Finally, we proved some results on image and preimage under homomorphism and anti-homomorphism on normal level sub HX group of a bipolar fuzzy normal HX subgroup.

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